Calculus II - Day 15

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Goals for today

- ~~Survive~~

Expression inside integral	**x-substitution**	**New expression**	**dx expression**
$a^2 - x^2$	$x = a\sin\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$a^2\cos^2\theta$	$dx = a\cos\theta\, d\theta$
$a^2 + x^2$	$x = a\tan\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2\sec^2\theta$	$dx = a\sec^2\theta\, d\theta$
$x^2 - a^2$	$x = a\sec\theta$ $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$a^2\tan^2\theta$	$dx = a\sec\theta\tan\theta\, d\theta$

Example

$$\int_{1.5}^{3} \frac{\sqrt{9-x^2}}{x^2}\, dx$$

Substitute $x = 3\sin\theta$, so $dx = 3\cos\theta\, d\theta$.
When $x = 3$, we have:

$$3 = 3\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \arcsin(1) = \frac{\pi}{2}$$

When $x = 1.5$, we get:

$$1.5 = 3\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Rewrite with new bounds:

$$= \int_{\pi/6}^{\pi/2} \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \frac{3\cos\theta}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \frac{9\cos^2\theta}{9\sin^2\theta}\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \cot^2\theta\, d\theta$$

Using the identity $\cot^2\theta = \csc^2\theta - 1$:

$$= \int_{\pi/6}^{\pi/2} \left(\csc^2\theta - 1\right)\, d\theta$$

$$= \left[-\cot\theta - \theta\right]_{\pi/6}^{\pi/2}$$

Now, evaluate at the bounds:

$$= \left(-\cot\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) - \left(-\cot\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right)$$

Since $\cot\left(\frac{\pi}{2}\right) = 0$ and $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$, we have:

$$= \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right)$$

$$= -\frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6}$$

Simplifying the final answer:

$$= \frac{-3\pi + \pi + 6\sqrt{3}}{6} = \frac{-2\pi + 6\sqrt{3}}{6} = \frac{-\pi}{3} + \sqrt{3}$$

Another Example

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\, dx$$

Substitute $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$.

Changing Bounds:

When $x = 0$, $\theta = \arctan(0) = 0$. When $x = 1$, $\theta = \arctan(1) = \frac{\pi}{4}$.

Thus, we rewrite the integral:

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\, dx = \int_0^{\pi/4} \frac{1}{(1+\tan^2\theta)^{3/2}} \sec^2\theta\, d\theta$$

Since $1 + \tan^2\theta = \sec^2\theta$:

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^3\theta}\, d\theta = \int_0^{\pi/4} \frac{1}{\sec\theta}\, d\theta = \int_0^{\pi/4} \cos\theta\, d\theta$$

Now, integrate and evaluate:

$$= \sin\theta\Big|_0^{\pi/4} = \sin\left(\frac{\pi}{4}\right) - \sin(0) = \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

$$\boxed{\frac{1}{\sqrt{2}}}$$

Now, let's consider the same integral as an indefinite integral:

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx$$

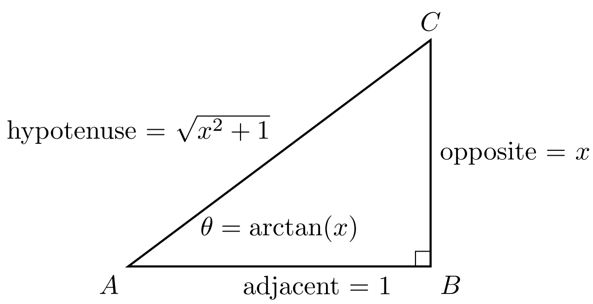
Substitute $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$.

$$= \int \frac{1}{(1+\tan^2\theta)^{3/2}} \sec^2\theta\,d\theta = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int \cos\theta\,d\theta$$

$$= \sin\theta + C$$

Since $x = \tan\theta$, we have $\theta = \arctan(x)$, so this becomes $\sin(\arctan(x)) + C$.

Now, we can use a right triangle to treat $\sin(\arctan(x))$ in a different form.



In this triangle, since $\theta = \arctan(x)$, $\tan\theta = \frac{x}{1}$. Therefore, we label the opposite side as x, the adjacent side as 1, and the hypotenuse as $\sqrt{x^2+1}$.

So, $\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$, which means:

$$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$$

Final answer:

$$\boxed{\frac{x}{\sqrt{1+x^2}} + C}$$

Another Example

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Substitute $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$.

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta \, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta} \, d\theta$$

Rewrite using trigonometric identities:

$$= \frac{1}{4}\int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta} \, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

At this point, there are two ways to proceed:

Trig substitution

$$= \frac{1}{4}\int \cot\theta\csc\theta \, d\theta$$

$$= -\frac{1}{4}\csc\theta + C$$

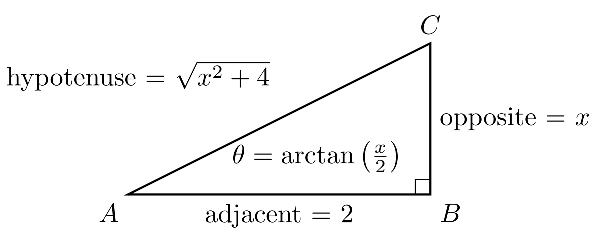
u-substitution

Let $u = \sin\theta$, then $du = \cos\theta \, d\theta$.

$$= \frac{1}{4}\int \frac{1}{u^2} \, du = -\frac{1}{4}u^{-1} + C$$

$$= -\frac{1}{4}\csc\theta + C$$

Now, convert back to x:



In this triangle, since $\theta = \arctan\left(\frac{x}{2}\right)$, $\tan\theta = \frac{x}{2}$. Thus, we label the opposite side as x, the adjacent side as 2, and the hypotenuse as $\sqrt{x^2+4}$.

Therefore, $\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2+4}}{x}$.

So,

$$-\frac{1}{4}\csc(\arctan(x/2)) + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

Final answer:

$$\boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$$

Example:

$$\int \frac{2x}{\sqrt{x^2+4}} \, dx$$

Don't be fooled—this looks like a trigonometric substitution, but we can instead use $u = x^2 + 4$.

A Secant Substitution Example

$$\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx$$

Using $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$.

Changing the bounds:

When $x = 1$:

$$1 = \sec\theta \Rightarrow 1 = \cos\theta$$

$$\theta = \arccos(1) = 0$$

When $x = 2$:

$$2 = \sec\theta \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Rewrite the integral with new bounds:

$$= \int_0^{\pi/3} \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,d\theta$$

Since $\sec^2\theta - 1 = \tan^2\theta$:

$$= \int_0^{\pi/3} \frac{1}{\tan\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} \sec\theta\,d\theta$$

Now, using the anti-derivative $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$:

$$= \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/3}$$

Note: This anti-derivative is useful to know.

Evaluating at the bounds:

$$= \ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \ln|\sec(0) + \tan(0)|$$

Since $\sec\left(\frac{\pi}{3}\right) = 2$ and $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, while $\sec(0) = 1$ and $\tan(0) = 0$:

$$= \ln|2+\sqrt{3}| - \ln|1|$$

$$= \ln(2+\sqrt{3})$$

$$\boxed{\ln(2+\sqrt{3})}$$

Example:

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$$

We need to "complete the square" to rewrite x^2+4x-5.

Complete the square:

$$x^2+4x-5 = (x^2+4x+4)-4-5 = (x+2)^2-9$$

Note: Completing the Square

How did we get to $(x+2)^2$?
To complete the square with x^2+ax+b:

$$x^2+ax+b = x^2+ax+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+b$$

$$= \left(x+\frac{a}{2}\right)^2+b-\left(\frac{a}{2}\right)^2$$

For this problem:

$$x^2+4x+4 = \left(x+\frac{4}{2}\right)^2 = (x+2)^2$$

With the completed square, we have:

$$x^2+4x-5 = (x+2)^2-9$$

Back to the problem.

$$\int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx$$

Using $u = x+2$, so $du = dx$,

$$u(1) = 3, \quad u(4) = 6$$

$$= \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du$$

Now let $u = 3\sec\theta$
$du = 3\sec\theta\tan\theta\,d\theta$.
When $u = 3$:

$$3 = 3\sec\theta \Rightarrow \sec\theta = 1$$

$$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$$

When $u = 6$:

$$6 = 3\sec\theta \Rightarrow \sec\theta = 2$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Rewrite the integral with new bounds:

$$= \int_0^{\pi/3} \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

Simplify:

$$= \int_0^{\pi/3} \frac{3\tan\theta}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3} 3\tan^2\theta\,d\theta$$

Rewrite $\tan^2\theta$ using $\tan^2\theta = \sec^2\theta - 1$:

$$= 3\int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$$

$$= [3\tan\theta - 3\theta]_0^{\pi/3}$$

Evaluating at bounds:

$$= (3\sqrt{3} - \pi) - (0 - 0)$$
$$= 3\sqrt{3} - \pi$$

$$\boxed{3\sqrt{3} - \pi}$$

Example:

$$\int \frac{x^2 + 2x + 1}{(15 - 2x - x^2)^{3/2}}\, dx$$

Substitute $u = x + 1$: $du = dx$

Rewrite $15 - 2x - x^2$ by completing the square:

$$15 - 2x - x^2 = 16 - (x+1)^2$$

$$= \int \frac{u^2}{(16 - u^2)^{3/2}}\, du$$

Now let $u = 4\sin\theta$, so $du = 4\cos\theta\, d\theta$

$$= \int \frac{(4\sin\theta)^2 \cdot 4\cos\theta}{(16\cos^2\theta)^{3/2}}\, d\theta$$

Simplify:

$$= \int \frac{16\sin^2\theta \cdot 4\cos\theta}{64\cos^3\theta}\, d\theta = \int \tan^2\theta\, d\theta$$

Use $\tan^2\theta = \sec^2\theta - 1$:

$$= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C$$

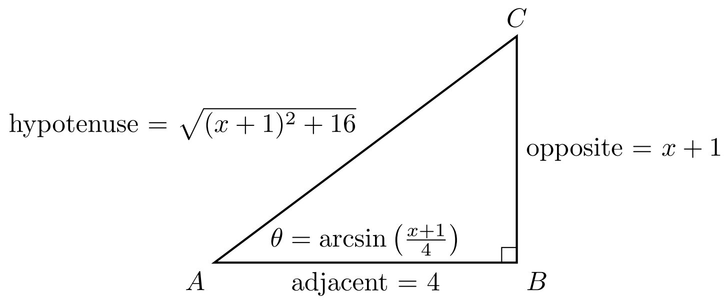
Substitute back $\theta = \arcsin\left(\frac{u}{4}\right)$

$$= \tan\left(\arcsin\left(\frac{u}{4}\right)\right) - \arcsin\left(\frac{u}{4}\right) + C$$

Substitute $u = x + 1$:

$$= \tan\left(\arcsin\left(\frac{x+1}{4}\right)\right) - \arcsin\left(\frac{x+1}{4}\right) + C$$

To express $\tan(\arcsin((x+1)/4))$ using a right triangle, consider the following triangle setup:



From the triangle:

$$\tan\left(\arcsin\left(\frac{x+1}{4}\right)\right) = \frac{x+1}{\sqrt{16-(x+1)^2}}$$

Final answer:

$$\boxed{\frac{x+1}{\sqrt{16-(x+1)^2}} - \arcsin\left(\frac{x+1}{4}\right) + C}$$

Note

The professor apologizes, noting that this is indeed a standard difficulty for a trigonometric substitution. The class objective was, in fact, correct.